

# PROPERTIES OF SQUEEZING FUNCTIONS AND GEOMETRY OF BOUNDED DOMAINS

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**ABSTRACT.** In this article we continue the study of properties of squeezing functions and geometry of bounded domains. The limit of squeezing functions of a sequence of bounded domains is studied. We give comparisons of intrinsic positive forms and metrics on bounded domains in terms of squeezing functions. To study the boundary behavior of squeezing functions, we introduce the notions of (intrinsic) ball pinching radius, and give boundary estimate of squeezing functions in terms of these datum. Finally, we use these results to study geometric and analytic properties of some interesting domains, including planar domains, Cartan-Hartogs domains, and a strongly pseudoconvex Reinhardt domain which is not convex. As a corollary, all Cartan-Hartogs domains are homogenous regular, i.e., their squeezing functions admit positive lower bounds.

## 1. INTRODUCTION

In a recent work [4], the authors introduced the notion of squeezing functions to study geometric and analytic properties of bounded domains. The squeezing function of a bounded domain  $D$  is defined as follows:

**Definition 1.1.** Let  $D$  be a bounded domain in  $\mathbb{C}^n$ . For  $p \in D$  and an (open) holomorphic embedding  $f : D \rightarrow B^n$  with  $f(p) = 0$ , we define

$$s_D(p, f) = \sup\{r | B^n(0, r) \subset f(D)\},$$

and the squeezing number  $s_D(p)$  of  $D$  at  $p$  is defined as

$$s_D(p) = \sup_f \{s_D(p, f)\},$$

where the supremum is taken over all holomorphic embeddings  $f : D \rightarrow B^n$  with  $f(p) = 0$ ,  $B^n$  is the unit ball in  $\mathbb{C}^n$ , and  $B^n(0, r)$  is the ball in  $\mathbb{C}^n$  with center 0 and radius  $r$ . We call  $s_D$  the *squeezing function* on  $D$ .

An important property of squeezing functions is their invariance under biholomorphic transformations. Namely, if  $f : D_1 \rightarrow D_2$  is a holomorphic equivalence of two bounded domains, then  $s_{D_2} \circ f = s_{D_1}$ . Some other interesting properties of squeezing functions were established in [4]. For example, for each  $p \in D$ , there exists an extremal map realizing the supremum in Definition 1.1, and squeezing functions are continuous.

In the present paper, we continue to study squeezing functions and applications to geometry of bounded domains.

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We first consider squeezing functions on a sequence of domains. We prove that, for a sequence of increasing domains convergent to a bounded domain, the squeezing functions of these domains converge to the squeezing function of the limit domain. We also prove a weaker result for a sequence of decreasing domains.

A homogenous regular domain (introduced in [11]) is a bounded domain whose squeezing function is bounded below by a positive constant. By the famous Bers embedding (see e.g. [6]), Teichmüller spaces of compact Riemann surfaces are homogenous regular domains. In the past decade, comparisons of various intrinsic metrics on Teichmüller spaces were intensively studied (see e.g. [2][11][18]). The equivalence of certain intrinsic measures on Teichmüller spaces was proved in [15]. In [11], it was proved that the Bergman metric, the Kobayashi metric, and the Carathéodory metric on a homogenous regular domain are equivalent. Geometric and analytic properties of homogenous regular domains were systematically studied in [19], where the term homogenous regular domain was phrased as uniformly squeezing domain. In this paper, we modify the method in [15] and [19] to give comparisons of invariant positive forms and metrics on general bounded domains in terms of squeezing functions.

For a smoothly bounded planar domain  $D$ , we have proved in [4] that  $\lim_{z \rightarrow p} s_D(z) = 1$  for all  $p \in \partial D$ . In this paper, we try to generalize the basic idea in [4] to study boundary behavior of squeezing functions on bounded domains of higher dimensions. For this purpose, we introduce the notions of ball pinching radius and intrinsic ball pinching radius of a bounded domain at its boundary points. The intrinsic ball pinching radii of a domain is a function defined on its boundary which is invariant under biholomorphic transformations. With lower semi-continuity of these functions being established, we can estimate the boundary behavior of the squeezing function of a domain at a boundary point in terms of the intrinsic ball pinching radius at this boundary point. In particular, a bounded domain is homogenous regular if the ball pinching radius at any boundary point is positive.

It seems that the above results can be used as powerful tools to study geometric and analytic properties of bounded domains. The key point is to estimate lower bounds of squeezing functions near boundary points. In this paper, we will study some special domains as examples.

The first example is planar domains. Using the results on squeezing functions mentioned above, we can recover some results in one complex variable, namely, we prove that the Bergman metric, the Kobayashi metric, and the Carathéodory metric on a planar domain have the same increasing order near a smooth boundary point.

The second example is Cartan-Hartogs domains, which are certain Hartogs domains with classical bounded symmetric domains as bases. In [22], Yin proposed a problem whether all Cartan-Hartogs domains are homogenous regular. In this paper, we answer this question affirmatively. This provides a class of homogenous regular domains with weakly pseudoconvex smooth boundary. Consequently, we establish many good analytic and geometric properties of Cartan-Hartogs domains. For example, these domains are hyperconvex and have bounded geometry; various classical intrinsic metrics, as well as all the volume forms considered in §3, on these domains are equivalent. We also give a boundary estimate of squeezing functions of Thullen domains defined as  $\{(z_1, z_2) \in \mathbb{C}^2; |z_1|^{2k} + |z_2|^2 < 1\}$  for  $k > 0$ , which

are special Cartan-Hartogs domains. A detailed estimate of squeezing functions on general Cartan-Hartogs domains will appear in a separate work.

The third example is the Reinhardt domain defined by

$$\{(z_1, z_2) \in \mathbb{C}^2; \log^2 |z_1|^2 + \log^2 |z_2|^2 < 1\}.$$

It is a strongly pseudoconvex domain with smooth boundary that is not convex. We prove that this domain is homogenous regular. Though it is just a special example, the method here seems interesting and possible to be generalized to study general strongly pseudoconvex domains.

The rest of the paper is organized as follows. In §2, we study the limit of squeezing functions of a sequence of domains; in §3, we describe the comparisons of intrinsic positive forms and metrics in terms of squeezing functions; in §4, we introduce the notion of ball pinching radius and intrinsic ball pinching radius, and give an estimate of boundary behavior of squeezing functions in terms of these datum; and in the final §5, we use the results in previous sections to study properties of some interesting domains.

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## 2. SQUEEZING FUNCTIONS ON LIMIT DOMAINS

In this section, we consider the relation between the limit of squeezing functions of a sequence of domains and the squeezing function of the limit domain. For a sequence of increasing domains, we have the following

**Theorem 2.1.** *Let  $D \subset \mathbb{C}^n$  be a bounded domain and  $D_k \subset D$  ( $k \in \mathbb{N}$ ) be a sequence of domains such that  $\cup_k D_k = D$  and  $D_k \subset D_{k+1}$  for all  $k$ . Then, for any  $z \in D$ ,  $\lim_{k \rightarrow \infty} s_{D_k}(z) = s_D(z)$ .*

*Proof.* By the existence of extremal maps w.r.t squeezing functions (see Theorem 2.1 in [4]), for each  $k$ , there is an injective holomorphic map  $f_k : D_k \rightarrow B^n$  such that  $f_k(z) = 0$  and  $B^n(0, s_{D_k}(z)) \subset f_k(D_k)$ . By Montel's theorem, we may assume the sequence  $f_k$  converges uniformly on compact subsets of  $D$  to a holomorphic map  $f : D \rightarrow \mathbb{C}^n$ .

We first prove that  $f$  is injective. Assume  $z \in D_{k_0}$  for some  $k_0 > 0$ , then it is clear that

$$s_{D_k}(z) \geq \frac{d(z, \partial D_k)}{\text{diam}(D_k)} \geq \frac{d(z, \partial D_{k_0})}{\text{diam}(D)}$$

for  $k > k_0$ . So there is a  $\delta > 0$  such that  $B^n(0, \delta) \subset f_k(D_k)$  for all  $k > k_0$ . Set  $g_k = f_k^{-1}|_{B^n(0, \delta)} : B^n(0, \delta) \rightarrow D$ . By Cauchy's inequality,  $|\det(dg_k(0))|$  is bounded above uniformly for all  $k > k_0$  by a positive constant. Hence there exists a constant  $c > 0$ , such that  $|\det(df_k(z))| > c$  for all  $k > k_0$ . This implies  $\det(df(z)) \neq 0$ . So the injectivity of  $f$  follows from Lemma 2.3 in [4] and the generalized Rouché's theorem (Theorem 3 in [12]).

Since  $f$  is injective, it is an open map (see e.g. Theorem 8.5 in [5]). On the other hand, it is clear that  $f(D) \subset \overline{B^n}$ . So we have  $f(D) \subset B^n$ .

We now prove that  $s_D(z) \geq \limsup_k s_{D_k}(z)$ . Let  $s_{D_{k_i}}$  be a subsequence such that  $\lim_{k_i \rightarrow \infty} s_{D_{k_i}}(z) = \limsup_k s_{D_k}(z) = r$ , then, as explained above, we have  $r > 0$ . Let  $\epsilon > 0$  be an arbitrary positive number less than  $r$ , then  $B^n(0, r - \epsilon) \subset f_k(D_{k_i})$  for  $k_i$  large enough. Set  $h_{k_i} = f_{k_i}^{-1}|_{B^n(0, r - \epsilon)}$ , then  $\lim_{k_i \rightarrow \infty} |\det(dh_{k_i}(z))| = |\det(df^{-1}(0))| \neq 0$ . By the argument mentioned above,  $h := \lim_{k_i} h_{k_i}$  is injective and hence  $h(B^n(0, r - \epsilon)) \subset D$ . This implies  $f(h(w))$  make sense for all  $w \in B^n(0, r - \epsilon)$ . It is clear that  $f(h(w)) = w$  for all  $w \in B^n(0, r - \epsilon)$ . So  $B^n(0, r - \epsilon) \subset f(D)$  and  $s_D(z) \geq r - \epsilon$ . Since  $\epsilon$  is arbitrary, we get  $s_D(z) \geq \limsup_k s_{D_k}(z)$ .

Finally, we prove that  $s_D(z) \leq \liminf_k s_{D_k}(z)$ . Let  $s_{D_{k'_i}}$  be a subsequence such that  $\lim_{k'_i \rightarrow \infty} s_{D_{k'_i}}(z) = \liminf_k s_{D_k}(z)$ . By the existence of extremal map, there exists an injective holomorphic map  $\varphi : D \rightarrow B^n$  such that  $\varphi(z) = 0$  and  $B^n(0, s_D(z)) \subset \varphi(D)$ . For arbitrary  $0 < \epsilon < s_D(z)$ , by assumption,  $\varphi^{-1}(B^n(0, s_D(z) - \epsilon)) \subset D_{k'_i}$  for  $k'_i$  large enough. So, for  $k'_i$  large enough, we have  $s_{D_{k'_i}}(z) \geq s_D(z) - \epsilon$ . This implies  $s_D(z) - \epsilon \leq \lim_{k'_i \rightarrow \infty} s_{D_{k'_i}}(z)$ . Since  $\epsilon$  is arbitrary, we get  $s_D(z) \leq \liminf_k s_{D_k}(z)$ .  $\square$

For a sequence of decreasing domains, we have

**Theorem 2.2.** *Let  $D \subset \mathbb{C}^n$  be a bounded domain and  $D_k \supset D$  ( $k \in \mathbb{N}$ ) be a sequence of domains such that  $\cap_k D_k = D$  and  $D_{k+1} \subset D_k$  for all  $k$ . Then, for any  $z \in D$ ,  $s_D(z) \geq \limsup_k s_{D_k}(z)$ .*

*Proof.* For each  $k$ , let  $f_k : D_k \rightarrow B^n$  an injective holomorphic map such that  $f_k(z) = 0$  and  $B^n(0, s_{D_k}(z)) \subset f_k(D_k)$ . By Montel's theorem, we may assume  $\lim_k f_k = f$  exists and give a holomorphic map from  $D$  to  $\mathbb{C}^n$ . By the same argument as in proof of Theorem 2.1, we see that  $f$  is injective and  $f(D) \subset B^n$ .

Without loss of generality, we assume  $\lim_k s_{D_k}(z) = r$ . Then, for any  $\epsilon > 0$ ,  $B^n(0, r - \epsilon) \subset f_k(D_k)$  for  $k$  large enough. Set  $g_k = f_k^{-1}|_{B^n(0, r - \epsilon)} : B^n(0, r - \epsilon) \rightarrow D_k$ . We can assume  $g_k$  converges uniformly on compact subsets of  $B^n(0, r - \epsilon)$  to a holomorphic map  $g : B^n(0, r - \epsilon) \rightarrow \mathbb{C}^n$ . Similarly, one can show that  $g$  is injective and hence open. On the other hand, by assumption, it is clear that  $g(B^n(0, r - \epsilon)) \subset \cap_{k \geq 1} \overline{D_k}$ . Hence  $g(B^n(0, r - \epsilon)) \subset \cap_{k \geq 1} D_k = D$ . This implies that  $f(g(w))$  makes sense for all  $w \in B^n(0, r - \epsilon)$ . It is clear that  $f(g(w)) = w$  for all  $w \in B^n(0, r - \epsilon)$ . So  $B^n(0, r - \epsilon) \subset f(D)$  and  $s_D(z) \geq r - \epsilon$ . Since  $\epsilon$  is arbitrary, we get  $s_D(z) \geq r = \lim_k s_{D_k}(z)$ .  $\square$

The following example shows that the strict inequality in Theorem 2.2 is possible:

**Example 2.1.** *Let  $D = \{(z_1, z_2) | 0 < |z_2| < |z_1| < 1\}$  be the Hartogs triangle in  $\mathbb{C}^2$ . For a positive number  $\epsilon$  (small enough), we define a domain  $V_\epsilon$  in  $\mathbb{C}^2$  as*

$$V_\epsilon = \{(z_1, z_2) | 0 < |z_1| < 1, 0 < |z_2| < \epsilon\}.$$

*Set  $D_\epsilon = D \cup V_\epsilon$ . Let  $z^j = (z_1^j, z_2^j)$  be a sequence of points in  $D$  satisfying the conditions  $|z_1^j| \leq (1 + \frac{1}{j})|z_2^j|$  and  $|z_2^j| > a$  for all  $j$ , where  $a > 0$  is a fixed constant. Then we have*

- 1).  $\lim_{j \rightarrow \infty} s_{D_\epsilon}(z^j) = 0$  uniformly with respect to  $\epsilon$ , and
- 2). there exists a positive constant  $c$ , such that  $s_D(z^j) \geq c$  for all  $j$ .

*Proof.* 1) By the Riemann's removable singularity theorem and Hartogs's extension theorem, the Carathéodory metric  $C_{D_\epsilon}$  on  $D_\epsilon$  is given by the restriction on  $D_\epsilon$  of the Carathéodory metric on  $\Delta \times \Delta$ . Note that the Carathéodory metric on  $\Delta \times \Delta$  is continuous, it is clear that there exists a sequence of positive numbers  $r^j$  such that  $\lim_j r_j = 0$  and the balls, denoted by  $B_\epsilon(z^j, r^j)$ , in  $D_\epsilon$  centered at  $z^j$  with radius  $r^j$  with respect to  $C_{D_\epsilon}$ , are not relatively compact in  $D_\epsilon$  for all  $j$  and all  $\epsilon$  small enough. Assume  $f : D_\epsilon \rightarrow B^2$  is an injective holomorphic map such that  $f(z^j) = 0$  and  $B^2(0, s_{D_\epsilon}(z^j)) \subset f(D_\epsilon)$ . By the decreasing property of Carathéodory metric, we see that  $f(B_\epsilon(z^j, \sigma(\frac{s_{D_\epsilon}(z^j)}{2})))$  is relatively compact in  $f(D_\epsilon)$ , where  $\sigma : [0, 1) \rightarrow \mathbb{R}$  is the function defined as  $\sigma(x) = \ln \frac{1+x}{1-x}$ . Since  $f$  is injective, this implies  $s_{D_\epsilon}(z^j) \leq 2\sigma^{-1}(r^j)$  for all  $j$ . Hence  $\lim_{j \rightarrow \infty} s_{D_\epsilon}(z^j)$  tends to 0 uniformly w.r.t  $\epsilon$ .

2) The map  $\varphi(z_1, z_2) = (z_1, \frac{z_2}{z_1})$  gives a holomorphic isomorphism from  $D$  to  $\Delta^* \times \Delta^*$ . Denote  $\varphi(z^j)$  by  $(w_1^j, w_2^j)$ , then  $|w_1^j|, |w_2^j| > a$ . Note that the squeezing function on  $\Delta^*$  is given by  $s_{\Delta^*}(z) = |z|$  (see Corollary 7.2 in [4]), so we have  $s_{\Delta^* \times \Delta^*}(w_1^j, w_2^j) \geq \frac{\sqrt{2}}{2}a$  for all  $j$ . By the holomorphic invariance of squeezing functions, we get  $s_D(z^j) \geq \frac{\sqrt{2}}{2}a$  for all  $j$ .  $\square$

### 3. COMPARISON OF INTRINSIC FORMS AND METRICS

In this section, we give comparisons of intrinsic positive forms and metrics on bounded domains in terms of squeezing functions.

#### 3.1. Comparison of systems of positive forms with decreasing property.

Let  $V$  be a complex vector space and  $V^*$  be its dual. Let  $e_1, \dots, e_n$  be a basis of  $V$  and  $e_1^*, \dots, e_n^*$  be the dual basis of  $V^*$ . An  $(n, n)$ -form  $u \in \bigwedge^{n,n} V^*$  is called positive (or strictly positive) if  $u = \lambda i e_1^* \wedge \bar{e}_1^* \wedge \dots \wedge i e_n^* \wedge \bar{e}_n^*$  for some  $\lambda \geq 0$  (or  $\lambda > 0$ ). Since  $V$  has a canonical orientation, the definition is independent of the choice of basis of  $V$ . Generally, following [3], we call an element  $u \in \bigwedge^{p,p} V^*$  (strictly) positive if

$$u \wedge i \xi_1 \wedge \bar{\xi}_1 \wedge \dots \wedge i \xi_{n-p} \wedge \bar{\xi}_{n-p}$$

is a (strictly) positive  $(n, n)$ -form on  $V$  for any linearly independent  $\xi_i \in V^*, 1 \leq i \leq n - p$ . For  $u, v \in \bigwedge^{p,p} V^*$ , we define  $u \geq v$  ( $u > v$ ) if  $u - v$  is positive (strictly positive). It turns out that  $u \in \bigwedge^{p,p} V^*$  is (strictly) positive if and only if, for any  $p$ -dimensional vector subspace  $W$  of  $V$ , the restriction  $u|_W$  of  $u$  on  $W$  is (strictly) positive.

Now let  $X$  be a complex manifold and  $u$  a  $(p, p)$ -form on  $X$  (whose coefficients are not necessarily continuous). We call  $u$  (strictly) positive if it is (strictly) positive pointwise. Two positive  $(p, p)$ -forms  $u$  and  $v$  are called equivalent if  $\frac{1}{c}v \leq u \leq cv$  for some positive number  $c \geq 1$ .

By a *system of positive  $(p, p)$ -forms*  $\mathcal{F}$  with decreasing property, we mean attaching a strictly positive  $(p, p)$ -form  $\mathcal{F}_D$  to each bounded domain  $D$ , such that  $\mathcal{F}_{D_1} \geq f^*(\mathcal{F}_{D_2})$  for bounded domains  $D_1, D_2$  and any holomorphic mapping  $f : D_1 \rightarrow D_2$ .

Let  $\mathcal{F}$  and  $\mathcal{G}$  be two systems of positive  $(p, p)$ -forms, the pinching function  $P_{\mathcal{F}\mathcal{G}} : (0, 1) \rightarrow \mathbb{R}$  is defined as

$$P_{\mathcal{F}\mathcal{G}}(r) := \inf\{\lambda > 0 | \mathcal{F}_{B^n_r}(0) \leq \lambda \mathcal{G}_{B^n}(0)\},$$

where  $B_r^n$  denotes the ball in  $\mathbb{C}^n$  with center 0 and radius  $r$ . Then  $0 < P_{\mathcal{F}\mathcal{G}}(r) < \infty$  for  $r \in (0, 1)$ . By the decreasing property, it is clear that  $P_{\mathcal{F}\mathcal{G}}(r)$  is decreasing on  $(0, 1)$ .

**Theorem 3.1.** *Let  $D$  be a bounded domain,  $\mathcal{F}$  and  $\mathcal{G}$  as above. Then*

$$\frac{1}{P_{\mathcal{G}\mathcal{F}}(s_D(z))} \mathcal{G}_D(z) \leq \mathcal{F}_D(z) \leq P_{\mathcal{F}\mathcal{G}}(s_D(z)) \mathcal{G}_D(z)$$

*holds for any  $z \in D$ . In particular, if  $D$  is homogenous regular and  $s_D(z) \geq c > 0$ , then*

$$\frac{1}{P_{\mathcal{G}\mathcal{F}}(c)} \mathcal{G}_D(z) \leq \mathcal{F}_D(z) \leq P_{\mathcal{F}\mathcal{G}}(c) \mathcal{G}_D(z), z \in D$$

*and hence  $\mathcal{F}_D$  and  $\mathcal{G}_D$  must be equivalent.*

*Proof.* For any  $z \in D$ , denote  $s_D(z)$  by  $r$  for simplicity. By the existence of extremal maps, there is an open holomorphic embedding  $f : D \rightarrow B^n$  such that  $f(z) = 0$  and  $B^n(r) \subset f(D)$ . By the decreasing property, we have

$$\begin{aligned} f^* \mathcal{F}_{B^n}(0) &\leq \mathcal{F}_D(z) \leq f^* \mathcal{F}_{B_r(0)}, \\ f^* \mathcal{G}_{B^n}(0) &\leq \mathcal{G}_D(z) \leq f^* \mathcal{G}_{B_r(0)}. \end{aligned}$$

The above two inequalities imply

$$\frac{1}{P_{\mathcal{G}\mathcal{F}}(s_D(z))} \mathcal{G}_D(z) \leq \mathcal{F}_D(z) \leq P_{\mathcal{F}\mathcal{G}}(s_D(z)) \mathcal{G}_D(z).$$

In particular, since  $P_{\mathcal{F}\mathcal{G}}$  and  $P_{\mathcal{G}\mathcal{F}}$  are decreasing, we have

$$\frac{1}{P_{\mathcal{G}\mathcal{F}}(c)} \mathcal{G}_D(z) \leq \mathcal{F}_D(z) \leq P_{\mathcal{F}\mathcal{G}}(c) \mathcal{G}_D(z)$$

if  $s_D(z) \geq c$ . □

Let  $D$  be a domain in  $\mathbb{C}^n$ , then the *Carathéodory volume form* on  $D$  is defined to be the  $(n, n)$ -form

$$\tilde{\mathcal{M}}_D^C(z) = |\tilde{M}_D^C(z)| \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n,$$

where

$$|\tilde{M}_D^C(z)| = \sup\{|\det f'(z)|^2; f : D \rightarrow B^n \text{ holomorphic with } f(z) = 0\};$$

and the *Eisenman-Kobayashi volume form* is defined to be the  $(n, n)$ -form

$$\tilde{\mathcal{M}}_D^K(z) = |\tilde{M}_D^K(z)| \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n,$$

where

$$|\tilde{M}_D^K(z)| = \inf\{1/|\det f'(0)|^2; f : B^n \rightarrow D \text{ holomorphic with } f(0) = z\}.$$

The Carathéodory volume form and the Eisenman-Kobayashi volume form satisfy the decreasing property. If  $D$  is bounded, then  $\tilde{\mathcal{M}}_D^C$  and  $\tilde{\mathcal{M}}_D^K$  are two strictly positive  $(n, n)$ -forms on  $D$  (see e.g. [9]). So  $\tilde{\mathcal{M}}^C$  and  $\tilde{\mathcal{M}}^K$  are two systems of positive  $(n, n)$ -forms.

Let  $h$  be a norm on  $\mathbb{C}^n$ , and let  $B^n(h) := \{v \in \mathbb{C}^n | h(v) < 1\}$  be the unit ball with respect to  $h$ . Then the volume form of  $h$  is defined as

$$\frac{\text{vol}(B^n)}{\text{vol}(B^n(h))} \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n,$$

where  $\text{vol}(B^n)$  and  $\text{vol}(B^n(h))$  denote the Euclidean volumes of  $B^n$  and  $B^n(h)$  respectively. Note that the volume form of  $h$  is completely determined by  $h$ , and independent of the choice the original inner product on  $\mathbb{C}^n$ .

On a bounded domain  $D$ , the Kobayashi metric and the Carathéodory metric are nondegenerate, namely, they give norms on tangent spaces at all points of  $D$ . So we can define the volume forms of the Kobayashi metric and the Carathéodory metric on  $D$  and denote them by  $\mathcal{M}_D^K$  and  $\mathcal{M}_D^C$  respectively. They are strictly positive  $(n, n)$ -forms on  $D$ . Since the Kobayashi metric and the Carathéodory metric satisfy the decreasing property (see e.g. [8]), so do their volume forms. Hence  $\mathcal{M}^C$  and  $\mathcal{M}^K$  are two systems of positive  $(n, n)$ -forms. Here one should note that, in general, the Carathéodory volume form and the volume form of the Carathéodory metric are distinct, and the Eisenman-Kobayashi volume form and the volume form of the Kobayashi metric are distinct.

On the unit ball  $B^n$ , all the four volume forms defined above coincide. Let  $\mathcal{M}$  and  $\mathcal{M}'$  be any two of the four volume forms, i.e., the Carathéodory volume form, the Eisenman-Kobayashi volume form, the volume form of the Carathéodory metric, and the volume form of the Kobayashi metric, then it is easy to see that

$$P_{\mathcal{M}\mathcal{M}'}(r) = P_{\mathcal{M}'\mathcal{M}}(r) = \frac{1}{r^{2n}}.$$

By Theorem 3.1, we have

**Theorem 3.2.** *Let  $D$  be a bounded domain in  $\mathbb{C}^n$ ,  $\mathcal{M}$  and  $\mathcal{M}'$  as above. Then we have*

$$s_D^{2n}(z)\mathcal{M}'_D(z) \leq \mathcal{M}_D(z) \leq \frac{1}{s_D^{2n}(z)}\mathcal{M}'_D(z), z \in D.$$

*In particular, if  $D$  is homogenous regular and  $s_D(z) \geq c > 0$ , then*

$$c^{2n}\mathcal{M}'_D(z) \leq \mathcal{M}_D(z) \leq \frac{1}{c^{2n}}\mathcal{M}'_D(z), z \in D,$$

*and hence  $\mathcal{M}_D$  and  $\mathcal{M}'_D$  are equivalent.*

**3.2. Comparison of metrics with decreasing property.** A metric  $h$  on a bounded domain  $D$  is a map

$$h : D \times \mathbb{C}^n \rightarrow \mathbb{R}$$

(not necessarily continuous) such that, for any  $z \in D$ , the restriction  $h_z$  of  $h$  on  $\{z\} \times \mathbb{C}^n$  gives a norm on  $\mathbb{C}^n$ . In general, a metric  $h$  on  $D$  can not be represented by a strictly positive  $(1, 1)$ -form if it is not Hermitian. Similarly, for a system of decreasing metrics  $\mathcal{H}$ , we mean attaching each bounded domain  $D$  a metric  $\mathcal{H}_D$ , such that  $\mathcal{H}_{D_1} \geq f^*(\mathcal{H}_{D_2})$  for bounded domains  $D_1, D_2$  and any holomorphic mapping  $f : D_1 \rightarrow D_2$ . Given two systems of decreasing metrics  $\mathcal{H}$  and  $\mathcal{H}'$ , we can define a pinching function  $P_{\mathcal{H}\mathcal{H}'} : (0, 1) \rightarrow \mathbb{R}$  by setting

$$P_{\mathcal{H}\mathcal{H}'}(r) := \inf\{\lambda > 0 | \mathcal{H}_{B_r^n}(0) \leq \lambda \mathcal{H}'_{B^n}(0)\}.$$

Then the same argument as the proof of Theorem 3.1 leads to the following

**Theorem 3.3.** *Let  $D$  be a bounded domain, and  $\mathcal{H}$  and  $\mathcal{H}'$  be two systems of decreasing metrics. Then*

$$\frac{1}{P_{\mathcal{H}'\mathcal{H}}(s_D(z))}\mathcal{H}'_D(z) \leq \mathcal{H}_D(z) \leq P_{\mathcal{H}\mathcal{H}'}(s_D(z))\mathcal{H}'_D(z).$$

In particular, if  $D$  is homogenous regular and  $s_D(z) \geq c > 0$ , then, for any  $z \in D$ , we have

$$\frac{1}{P_{\mathcal{H}'\mathcal{H}}(c)}\mathcal{H}'_D(z) \leq \mathcal{H}_D(z) \leq P_{\mathcal{H}\mathcal{H}'}(c)\mathcal{H}'_D(z), z \in D,$$

and hence  $\mathcal{H}_D$  and  $\mathcal{H}'_D$  must be equivalent.

It is known that the Kobayashi metric  $\mathcal{H}^K$  and Carathéodory metric  $\mathcal{H}^C$  on bounded domains are Finsler metrics satisfying the decreasing property. They are coincide on the unit ball, and we have

$$P_{\mathcal{H}^K\mathcal{H}^C}(r) = P_{\mathcal{H}^C\mathcal{H}^K}(r) = \frac{1}{r}, r \in (0, 1).$$

It is also well known that the Carathéodory metric on a bounded domain is dominated by its Kobayashi metric. So a direct consequence of Theorem 3.3 is the following

**Corollary 3.4.** *Let  $D$  be a bounded domain. Then*

$$s_D(z)\mathcal{H}_D^K(z) \leq \mathcal{H}_D^C(z) \leq \mathcal{H}_D^K(z).$$

In particular, if  $D$  is homogenous regular and  $s_D(z) \geq c > 0$ , then, for any  $z \in D$ , we have

$$c\mathcal{H}_D^K(z) \leq \mathcal{H}_D^C(z) \leq \mathcal{H}_D^K(z), z \in D,$$

and hence  $\mathcal{H}_D^C$  and  $\mathcal{H}_D^K$  must be equivalent.

For a metric  $h$  on a bounded domain  $D$ , as explained in the above subsection, we can define the volume  $\mathcal{M}h$  of  $h$ , which is a strictly positive  $(n, n)$ -form on  $D$ . If there are two metrics  $h$  and  $h'$  on  $D$  satisfying the condition

$$a(z)h'(z) \leq h(z) \leq b(z)h'(z), z \in D,$$

where  $a$  and  $b$  are two continuous strictly positive functions on  $D$ , then the volume forms  $\mathcal{M}h$  and  $\mathcal{M}h'$  satisfy the comparison

$$(a(z))^{2n}\mathcal{M}h'(z) \leq \mathcal{M}h(z) \leq (b(z))^{2n}\mathcal{M}h'(z), z \in D.$$

In particular, if  $h$  and  $h'$  are equivalent, then  $\mathcal{M}h$  and  $\mathcal{M}h'$  are also equivalent.

We have shown in Theorem 3.2 that the volume forms of the Kobayashi metric and the Carathéodory metric on a homogenous regular domain are equivalent, and they are equivalent to the Carathéodory volume form and the Kobayashi volume form. We also see that, on a homogenous regular domain, the Kobayashi metric and the Carathéodory metric are equivalent. It is also known that they are equivalent to the Bergman metric and the Kähler-Einstein metric [11] [19]. As a consequence, we have

**Theorem 3.5.** *On a homogenous regular domain, the volume forms of the Kobayashi metric, the Carathéodory metric, the Bergman metric, and the Kähler-Einstein metric are equivalent, and they are equivalent to the Carathéodory and the Eisenman-Kobayashi volume forms.*

The equivalence of some of the above volume forms was established in [15] for Teichmüller spaces. In the following subsection, we will describe comparisons of the Kobayashi metric, the Bergman metric, and the Kähler-Einstein metric (if the domain considered is pseudoconvex) and their volume forms on a general bounded domain in terms of its squeezing function.



**3.3. Comparison of intrinsic metrics in terms of squeezing functions.** For a bounded domain  $D$ , we have got a comparison between its Carathéodory metric  $\mathcal{H}_D^C$  and Kobayashi metric  $\mathcal{H}_D^K$  in terms of its squeezing function in Corollary 3.4. The Bergman metric  $\mathcal{H}_D^B$ , which does not satisfy the decreasing property, is invariant under biholomorphic transformations. When  $D$  is pseudoconvex, it is well known that there is a unique complete Kähler-Einstein metric on  $D$ , denoted by  $\mathcal{H}_D^{KE}$ , with Ricci curvature normalized by  $-(n+1)$  [14], which is also invariant under biholomorphic transformations. In this section, we will give a comparison of the Kobayashi metric  $\mathcal{H}_D^K$  with the Bergman metric  $\mathcal{H}_D^B$  and, if  $D$  is pseudoconvex, the Kähler-Einstein metric  $\mathcal{H}_D^{KE}$  in terms of the squeezing function on  $D$ . The main result is

**Theorem 3.6.** *Let  $D$  be a bounded domain in  $\mathbb{C}^n$  and  $z \in D$ , and let  $s_D$  be the squeezing function on  $D$ . Then we have*

$$s_D(z)\mathcal{H}_D^K(z) \leq \mathcal{H}_D^B(z) \leq \frac{2^{n+2}\pi}{s_D^{n+1}(z)}\mathcal{H}_D^K(z). \quad (1)$$

If in addition  $D$  is pseudoconvex, then

$$\sqrt{\frac{2}{n}}s_D(z)\mathcal{H}_D^K(z) \leq \mathcal{H}_D^{KE}(z) \leq \left(\frac{n}{2s_D^2(z)}\right)^{(n-1)/2}\mathcal{H}_D^K(z). \quad (2)$$

*Remark 3.1.* If  $D$  is homogenous regular and  $s_D(z) \geq c$  for some constant  $c > 0$ , the above comparison, with  $s_D(z)$  replaced by  $c$ , was proved in [19]. In particular, the Bergman metric and the Kähler-Einstein metric on  $D$  are equivalent to the Kobayashi metric. As we will see, a slight modification of the method in [19] can be used to give the proof of Theorem 3.6.

*Proof.* (the proof of Theorem 3.6) We denote  $s_D(z)$  by  $r$  for simplicity. By the existence of extremal maps [4], there exists an open imbedding  $f : D \rightarrow B^n$  such that  $f(z) = 0$  and  $B(0, r) \subset f(D)$ . By the holomorphic invariance of these metrics considered, we may assume  $B^n(z, r) \subset D \subset B^n(z, 1)$ , where  $B^n(z, r)$  denotes the ball in  $\mathbb{C}^n$  with center  $z$  and radius  $r$ .

We first prove (1). Due to the estimates in [13], we know that  $\mathcal{H}_D^C(z) \leq \mathcal{H}_D^B(z)$ . By Corollary 3.4, we get

$$r\mathcal{H}_D^K(z) \leq \mathcal{H}_D^B(z). \quad (3)$$

It's known that (for example, see [8], page 189) the Bergman kernel  $K_D(z, \bar{z}) = \sup\{|f(z)|^2 : f \in L_h^2(D), \|f\|_{L^2} = 1\}$ , where  $L_h^2(D)$  is the space of square integrable holomorphic functions on  $D$ . Let  $f_z$  be a function that realizes the supremum. Then, for  $V \in T_z D$ , its norm  $\mathcal{H}_D^B(z, V)$  of  $V$  w.r.t the Bergman metric on  $D$  is given by

$$\mathcal{H}_D^B(z, V) = \frac{1}{|f_z(z)|} \sup_{g \in L_h^2(D), \|g\|_{L^2} = 1, g(z) = 0} |V(g)|. \quad (4)$$

In particular, we take  $V = \frac{\partial}{\partial z^i}$ , then

$$\mathcal{H}_D^B(z, \frac{\partial}{\partial z^i}) = \frac{\left|\frac{\partial}{\partial z^i} g_{i,z}(z)\right|}{|f_z(z)|},$$

where  $g_{i,z}$  is a holomorphic function realizing the supremum of in (4) with  $V = \frac{\partial}{\partial z^i}$ . By the mean value inequality and the Cauchy inequality, a similar computation as

in [19] shows that

$$|f_z(z)| \geq \sigma_n^{-1/2},$$

$$\left| \frac{\partial}{\partial z^i} g_{i,z}(z) \right| \leq \sigma_n^{-1/2} \frac{2^{n+2}\pi}{r^{n+1}},$$

where  $\sigma_n$  is the Euclidean volume of the unit ball in  $\mathbb{C}^n$ . Consequently, we obtain that

$$\mathcal{H}_D^B(z, \frac{\partial}{\partial z^i}) \leq \frac{2^{n+2}\pi}{r^{n+1}}. \quad (5)$$

By the decreasing property of the Kobayashi metric, we have

$$\mathcal{H}_D^K(z, \frac{\partial}{\partial z^i}) \geq \mathcal{H}_{B^n(z,1)}^K(z, \frac{\partial}{\partial z^i}) = 1. \quad (6)$$

Combine (5) and (6), we get

$$\mathcal{H}_D^B(z, \frac{\partial}{\partial z^i}) \leq \frac{2^{n+2}\pi}{r^{n+1}} \mathcal{H}_D^K(z, \frac{\partial}{\partial z^i}). \quad (7)$$

Then (1) is obtained by combining (3) and (7).

We now prove (2) in this theorem. Recall that the Kobayashi metric  $\mathcal{H}_{B^n(z,r)}^K$  on  $B^n(z,r)$  is a Kähler metric with constant holomorphic sectional curvature  $-4$  and constant Ricci curvature  $-\frac{n+1}{r^2}$ . For a vector  $v \in T_z(B^n(z,r))$ , its Kobayashi norm is given by  $\mathcal{H}_{B^n(z,r)}^K(z, v) = \|v\|/r$ , where  $\|v\|$  denotes the Euclidean norm of  $v$ . From Mok-Yau's schwarz lemma [14] and the decreasing property of Kobayashi metric, we have

$$\text{vol}(\mathcal{H}_D^{KE}(z)) \leq \frac{1}{r^{2n}} \text{vol}(\mathcal{H}_{B^n(z,r)}^K(z)) = \text{vol}(\mathcal{H}_{B^n(z,1)}^K(z)) \leq \text{vol}(\mathcal{H}_D^K(z)), \quad (8)$$

where, for a metric  $g$ ,  $\text{vol}(g)$  denotes the volume form of  $g$ . On the other hand, by Royden's Schwarz lemma [16], we have

$$\mathcal{H}_D^{KE}(z) \geq \sqrt{\frac{2}{n}} \mathcal{H}_{B^n(z,1)}^K(z) = \sqrt{\frac{2}{n}} r \mathcal{H}_{B^n(z,r)}^K(z) \geq \sqrt{\frac{2}{n}} r \mathcal{H}_D^K(z). \quad (9)$$

Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$  be the eigenvalues of  $\mathcal{H}_D^{KE}(z)$  with respect to  $\mathcal{H}_D^K(z)$ , then it follows from (8) that  $\prod_{j=1}^n \lambda_j \leq 1$ , and from (9) that  $\lambda_j \geq \frac{2r^2}{n}$ . Hence  $\lambda_1 \leq \left(\frac{n}{2r^2}\right)^{n-1}$ , and we conclude that

$$\sqrt{\frac{2}{n}} r \mathcal{H}_D^K(z) \leq \mathcal{H}_D^{KE}(z) \leq \left(\frac{n}{2r^2}\right)^{(n-1)/2} \mathcal{H}_D^K(z).$$

□

Let  $\mathcal{M}_D^K$  and  $\mathcal{M}_B^K$  be the volume forms of the Kobayashi metric and the Bergman metric on  $D$  respectively, and let  $\mathcal{M}_D^{KE}$  be the volume form of the Kähler-Einstein metric on  $D$  if  $D$  is pseudoconvex. Then a direct corollary of Theorem 3.6 is:

**Corollary 3.7.** *Let  $D$  be a bounded domain in  $\mathbb{C}^n$  and  $z \in D$ , let  $s_D$  be the squeezing function of  $D$ . Then we have*

$$s_D^{2n}(z) \mathcal{M}_D^K(z) \leq \mathcal{M}_D^B(z) \leq \left(\frac{2^{n+2}\pi}{s_D^{n+1}(z)}\right)^{2n} \mathcal{M}_D^K(z). \quad (10)$$

If in addition  $D$  is pseudoconvex, then

$$\left(\frac{2}{n}\right)^n s_D^{2n}(z) \mathcal{M}_D^K(z) \leq \mathcal{M}_D^{KE}(z) \leq \left(\frac{n}{2s_D^2(z)}\right)^{n(n-1)} \mathcal{M}_D^K(z). \quad (11)$$

#### 4. BOUNDARY ESTIMATES OF SQUEEZING FUNCTIONS

The main aim of this section is to study boundary behavior of squeezing functions on bounded domains. We first introduce the notion of ball pinching radius and intrinsic ball pinching radius of a bounded domain at its boundary points, and establish semi-continuity of these functions in §§4.1. In §§4.2, we give a boundary estimate of squeezing functions in terms of intrinsic ball pinching radius.

**4.1. Intrinsic ball pinching radius.** Let  $D$  be a bounded domain in  $\mathbb{C}^n$  and  $p \in \partial D$  be a boundary point of  $D$ . If  $D$  is  $C^2$ -smoothly bounded at  $p$  and contained in some ball in  $\mathbb{C}^n$  with boundary point  $p$ , then we define  $e_D(p)$  the minimum of the radii of balls with boundary point  $p$  that contain  $D$ . If  $D$  is not  $C^2$ -smoothly bounded at  $p$  or no ball with boundary point  $p$  can contain  $D$ , then we set  $e_D(p) = +\infty$ . The *ball pinching radius* of  $D$  at  $p$ , denoted by  $B_D(p)$ , is defined to be

$$B_D(p) := \sup_a \left\{ \frac{a}{e_D(p)} \right\}$$

if  $e_D(p) < +\infty$ , where the supremum is taken over all positive numbers  $a$  satisfying the condition: there exists a ball of radius  $a$  with boundary point  $p$  such that its intersection with some neighborhood of  $p$  in  $\mathbb{C}^n$  is contained in  $D$ . If  $e_D(p) = +\infty$ , we set  $B_D(p) = 0$ .

The *intrinsic ball pinching radius* of  $D$  at  $p$ , denoted by  $IB_D(p)$ , is defined as follows:

$$IB_D(p) := \sup_{D', f} \{ B_{D'}(p') \},$$

where the supremum is taken over  $(D', f)$  with condition:  $D'$  is a bounded domain in  $\mathbb{C}^n$ ,  $f$  is a biholomorphic map from  $D$  to  $D'$  such that  $f$  can be extended to a continuous map from  $D \cup \{p\}$  to  $D' \cup \{p'\}$ , where  $p'$  is a boundary point of  $D'$ .

We view  $B_D$  and  $IB_D$  as two functions defined on  $\partial D$ . It is clear that  $B_D \leq IB_D$ . By definition, we see that  $IB_D$  is invariant under biholomorphic transformations. More precisely, let  $D$  and  $D'$  be two bounded domains,  $p \in \partial D$  and  $p' \in \partial D'$ . If there exists a biholomorphic map  $f : D \rightarrow D'$  such that  $f$  can be extended to a continuous map from  $D \cup \{p\}$  to  $D' \cup \{p'\}$ , then  $IB_D(p) = IB_{D'}(p')$ .

By definition, we call  $p \in \partial D$  a *globally strongly convex* (g.s.c) boundary point of  $D$  if  $B_D(p) > 0$ , or equivalently  $e_D(p) < +\infty$ .

Our main aim in this section is to study the relation between (intrinsic) ball pinching radius and boundary behavior of squeezing functions.

The following Proposition gives some basic properties of the two functions:

**Proposition 4.1.** *Let  $D$  be a bounded domain. We have*

- 1). *both  $B_D$  and  $IB_D$  are lower semi-continuous on  $\partial D$ ;*
- 2). *for  $p \in \partial D$ , if  $B_D(p) > s$  for some constant  $s > 0$ , then there exists a neighborhood  $U$  of  $p$  in  $\mathbb{C}^n$  such that, for any  $q \in U \cap \partial D$ , the intersection of  $U$  and the ball of radius  $s \cdot e_D(q)$  with boundary point  $q$  is contained in  $D$ .*

The Proposition can be viewed as a result in differential topology and its proof will be given as an appendix at the end of the paper.

*Remark 4.1.* 1). Assume  $p \in \partial D$  with  $e_D(p) > 0$ . Let  $\rho$  be a local defining function of  $D$  near  $p$  with  $|\nabla \rho(p)| = 1$ . Let  $\lambda$  be the biggest eigenvalue of the restriction on  $T_p \partial D$  of the real Hessian of  $\rho$  at  $p$ . Then it is easy to see that  $B_D(p) = \frac{1/\lambda}{e_D(p)}$ ;  
 2). It is obvious that a  $C^2$ -smooth boundary point  $p$  of  $D$  is g.s.c if and only if the real Hessian of the local defining function of  $D$  near  $p$  is positive definite on  $T_p(\partial D)$  and  $T_p(\partial D) \cap \bar{D} = \{p\}$ .

**4.2. Boundary behavior of squeezing functions.** In this subsection, we give estimates of boundary behavior of squeezing functions on bounded domains in terms of their intrinsic ball pinching radius introduced in the above subsection.

For the unit disc  $\Delta$  in  $\mathbb{C}$ , it is well known that all geodesic balls in  $\Delta$  with respect to the Poincaré metric are discs. This holds since all automorphisms of  $\Delta$  are fractional linear transformations and all fractional linear transformations map discs to discs. But it is not the case in higher dimensions. In general, a geodesic ball of the unit ball in  $\mathbb{C}^n$  with  $n > 1$  with respect to the Kobayashi metric is not a ball. In fact, we have the following Proposition 4.2, which will be used in our discussion of squeezing functions.

**Proposition 4.2.** *Let  $\Omega_\rho \subset B^n$  ( $n > 1$ ) be the ball centered at  $(1 - \rho, 0, \dots, 0)$  with radius  $\rho < 1$ . For  $0 < r < 1$ , denote  $(r, 0, \dots, 0) \in B^n$  by  $\mathbf{r}$ . Then, for  $r > \max\{1/2, 1 - 2\rho\}$ , the Kobayashi distance  $K_{B^n}(\mathbf{r}, \partial\Omega_\rho)$  on  $B^n$  from  $\mathbf{r}$  to  $\partial\Omega_\rho$  is given by*

$$K_{B^n}(\mathbf{r}, \partial\Omega_\rho) = \log \frac{1 + \sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}}{1 - \sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}}.$$

*In particular,  $K_{B^n}(\mathbf{r}, \partial\Omega_\rho)$  tends to  $\log \frac{1+\sqrt{\rho}}{1-\sqrt{\rho}}$  as  $r$  tends to 1.*

*Proof.* Let  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  be two point in  $\mathbf{B}^n$ . Then the Kobayashi distance of these two points is

$$K_{B^n}(z, w) = \log \left( \frac{|1 - w \cdot \bar{z}| + \sqrt{|z - w|^2 + |z \cdot \bar{w}|^2 - |z|^2 |w|^2}}{|1 - w \cdot \bar{z}| - \sqrt{|z - w|^2 + |z \cdot \bar{w}|^2 - |z|^2 |w|^2}} \right),$$

where  $z \cdot \bar{w} = \sum_k z_k \bar{w}_k$ . Note that  $\partial\Omega_\rho$  is given by

$$\partial\Omega_\rho = \left\{ z \in \mathbb{C}^n : |z_1 - (1 - \rho)|^2 + \sum_{k=2}^n |z_k|^2 = \rho^2 \right\}.$$

For  $z \in \partial\Omega_\rho$ ,  $z \neq (1, 0, \dots, 0)$ , a direct computation shows that

$$K_{B^n}(\mathbf{r}, z) = \log \frac{1 + \sqrt{\varphi(z_1)}}{1 - \sqrt{\varphi(z_1)}},$$

where

$$\varphi(z_1) = \frac{|r - z_1|^2 + (1 - r^2)(\rho^2 - |z_1 - 1 + \rho|^2)}{|1 - r\bar{z}_1|^2}.$$

We need to compute the minimal value of  $\varphi(z_1)$ . For  $z_1 = x + iy$ , we have the identity

$$|r - z_1|^2 + (1 - r^2)(\rho^2 - |z_1 - 1 + \rho|^2) = |1 - rz_1|^2 + 2(r^2 - 1)(1 - \rho)(1 - x).$$

So

$$\varphi(z_1) = 1 + \frac{2(r^2 - 1)(1 - \rho)(1 - x)}{|1 - r\bar{z}_1|^2}.$$

It is easy to see that

$$\varphi(z_1) = 1 - \frac{2(1 - r^2)(1 - \rho)(1 - x)}{|1 - r\bar{z}_1|^2} \geq 1 - \frac{2(1 - r^2)(1 - \rho)(1 - x)}{(1 - rx)^2},$$

and the equality holds if and only if  $y = 0$ . Let

$$\psi(x) = \frac{(1 - x)}{(1 - rx)^2}, \quad x \in (0, 1).$$

The maximal value of  $\psi(x)$  is obtained at  $x = 2 - \frac{1}{r}$  (note that here  $r$  tends to 1) and  $\psi(2 - \frac{1}{r}) = \frac{1}{4r(1-r)}$ . Then

$$\varphi(z_1) \geq 1 - \frac{(1 + r)(1 - \rho)}{2r},$$

and the equality holds if and only if  $x = 2 - \frac{1}{r}$  and  $y = 0$ . So, for  $z \in \partial\Omega_\rho$ , we have

$$K_{B^n}(\mathbf{r}, z) \geq \log \frac{1 + \sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}}{1 - \sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}}$$

and hence

$$K_{B^n}(\mathbf{r}, \partial\Omega_\rho) = \log \frac{1 + \sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}}{1 - \sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}},$$

which tends to  $\log \frac{1+\sqrt{\rho}}{1-\sqrt{\rho}}$  as  $r$  tends to 1.  $\square$

With Proposition 4.2, we can prove the following

**Proposition 4.3.** *Let  $D$  be a bounded domain in  $\mathbb{C}^n$  and  $p$  be a boundary point of  $D$ . If  $B_D(p) > \rho$  for a positive number  $\rho$ . Then there is a neighborhood  $U$  of  $p$  such that*

$$s_D(z) \geq \sqrt{1 - \frac{(2 - \delta(z)/e_D(p))(1 - \rho)}{2(1 - \delta(z)/e_D(p))}}$$

for all  $z \in U \cap D \cap N$ , where  $\delta(z) = d(z, \partial D)$  is the boundary distance function  $e_D(p)$  is defined as in §§4.1, and  $N$  is the normal line of  $\partial D$  at  $p$ . In particular, for  $\rho \rightarrow B_D(p)$ , we have

$$\liminf_{N \ni z \rightarrow p} s_D(z) \geq \sqrt{B_D(p)}.$$

*Proof.* For  $0 \leq x < 1$ , we set  $\sigma(x) = \log \frac{1+x}{1-x}$ , which is a strictly increasing function. For  $z \in B^n$ , the Kobayashi distance  $K_{B^n}(z, 0)$  from  $z$  to 0 is  $\sigma(|z|)$ .

We may assume  $p = (1, 0, \dots, 0)$  and  $e_D(p) = 1$ , and assume the normal line  $N$  of  $\partial D$  at  $p$  is the line  $\{(x, 0, \dots, 0) | x \in \mathbb{R}\} \subset \mathbb{C}^n$ .

By assumption, there exists a ball  $\Omega$  of radius  $\rho$  and a neighborhood  $U$  of  $p$  such that  $\partial\Omega$  is tangent to  $\partial B^n$  at  $p$  and  $\Omega \cap U \subset D$ . Since the Kobayashi metric on  $B^n$  is complete, for any positive  $s > 0$ , the Kobayashi geodesic ball in  $B^n$  centered at  $(r, 0, \dots, 0)$  with radius  $s$  must be contained in  $U \cap B^n$  for  $r$  approaching to 1 enough. By Proposition 4.2, for  $1 - r$  small enough, the Kobayashi geodesic ball

centered at  $\mathbf{r} = (r, 0, \dots, 0)$  with radius  $\sigma(\sqrt{1 - \frac{(1+r)(1-\rho)}{2r}})$  is contained in  $\Omega \cap U \subset D$ . Note that  $B^n$  is homogenous, there is a biholomorphic equivalence  $F \in \text{Aut}(B^n)$  such that  $F(\mathbf{r}) = 0$ . By the holomorphic invariance of the Kobayashi metric,  $F(D)$  contains a Kobayashi geodesic ball centered at 0 with radius  $\sigma(\sqrt{1 - \frac{(1+r)(1-\rho)}{2r}})$ , which is the Euclidean ball centered at 0 with radius (w.r.t the Euclidean metric)  $\sqrt{1 - \frac{(1+r)(1-\rho)}{2r}}$ . Note that, for  $U$  small enough, we have  $\delta(\mathbf{r}) = 1 - r$  for any  $\mathbf{r} \in U$ . So we get

$$s_D(\mathbf{r}) \geq \sqrt{1 - \frac{(2 - \delta(\mathbf{r}))(1 - \rho)}{2(1 - \delta(\mathbf{r}))}}.$$

Let  $r$  tends to 1, we get

$$\liminf_{r \rightarrow 1} s_D(\mathbf{r}) \geq \sqrt{\rho}.$$

□

*Remark 4.2.* From the proof, we see that the neighborhood  $U$  appearing in the above Proposition is taken to satisfy two conditions: the intersection of  $U$  and  $\Omega$  appearing in the proof is contained in  $D$ , and for any  $z \in U \cap N$ , the Euclidean distance  $\delta(z)$  from  $z$  to  $\partial D$  is given by the length of the normal line segment from  $z$  to  $p$ . By the theory of tubular neighborhood in differential topology (see e.g. [7]), the second condition can be satisfied if  $U$  is small enough.

Combing Proposition 4.1 and Proposition 4.3, we can get the following theorem, which is the main result of this section:

**Theorem 4.4.** *Let  $D$  be a bounded domain in  $\mathbb{C}^n$  and  $p \in \partial D$ . Then*

$$\liminf_{z \rightarrow p} s_D(z) \geq \sqrt{IB_D(p)}$$

for all  $p \in \partial D$ .

*Proof.* We assume  $IB_D(p) > 0$ . Note that squeezing functions are invariant under biholomorphic transformations. So, taking a biholomorphic transformation if necessary, we can assume  $B_D(p) > \rho$ . By the theory of tubular neighborhood in differential topology (see e.g. [7]), there is a neighborhood  $U$  of  $p$  in  $\mathbb{C}^n$  such that, for any  $z \in U \cap D$ , the Euclidean distance  $\delta(z)$  from  $z$  to  $\partial D$  is given by the length of the normal line segment from  $z$  to the boundary. By Proposition 4.1, we can take  $U$  small enough such that the intersection of  $U$  and the ball of radius  $\rho \cdot e_D(q)$  with boundary point  $q$  is contained in  $D$  for any  $q \in U \cap \partial D$ . Set  $C = \inf_{q \in U \cap \partial D} \{e_D(q)\}$ . By Proposition 4.3 and the remark after it, we get

$$s_D(z) \geq \sqrt{1 - \frac{(2 - \delta(z)/C)(1 - \rho)}{2(1 - \delta(z)/C)}}$$

for all  $z \in U \cap D$ . Let  $z \rightarrow p$ , we get the estimate. □

A direct consequence of Theorem 4.4 is the following

**Corollary 4.5.** *Let  $D$  be a bounded domain and  $IB_D(p) > 0$  for all  $p \in \partial D$ , then  $D$  is a homogenous regular domain.*

Let  $D \subset \mathbb{C}^n$  be a bounded domain with  $C^2$ -smooth boundary. Then it is clear that there is a ball, say  $B$ , in  $\mathbb{C}^n$  such that  $D \subset B$  and  $\partial B \cap \partial D$  contains at least two points. Note that  $B_D(p) > 0$  for any  $p \in \partial B \cap \partial D$ . By Theorem 4.4, we have the following

**Corollary 4.6.** *Let  $D$  be a bounded domain with  $C^2$ -smooth boundary. Then there exist at least two points, say  $p_1$  and  $p_2$ , in  $\partial D$  such that  $\liminf_{z \rightarrow p_i} s_D(z_i) > 0$ ,  $i = 1, 2$ .*

It is known that any strongly convex bounded domain with smooth boundary is homogenous regular [19]. In this paper, we can say more about this. Let  $D \subset \mathbb{C}^n$  be a strongly convex bounded domain with  $C^2$ -smooth boundary, then  $B_D(p) > 0$  for all  $p \in \partial D$ . By Proposition 4.1, we see that  $\liminf_{p \in \partial D} B_D(p) > 0$ . By Theorem 4.4, we get the following

**Corollary 4.7.** *Let  $D \subset \mathbb{C}^n$  be a strongly convex bounded domain with  $C^2$ -smooth boundary, and let  $\rho = \liminf_{p \in \partial D} B_D(p) > 0$ . Then we have*

$$\liminf_{z \rightarrow \partial D} s_D(z) \geq \sqrt{\rho}.$$

*In particular, by the continuity of  $s_D$  (see Theorem 3.1 in [4]),  $D$  must be a homogenous regular domain.*

Let  $D$  be a bounded domain in  $\mathbb{C}^n$  and  $p \in \partial D$ . Assume there is a ball  $B$  and a neighborhood  $V$  of  $p$  in  $\mathbb{C}^n$  such that  $D \subset B$  and  $\partial D \cap V = \partial B \cap V$ . Then, by Theorem 4.4, we have  $\lim_{z \rightarrow q} s_D(z) = 1$  for any  $q \in \partial D \cap V$ . On the other hand, we conjecture as follows that essentially the inverse of this result is true. As mentioned in the introduction, for a planar domain  $D$  with smooth boundary, we have  $\lim_{z \rightarrow \partial D} s_D(z) = 1$ . However, it seems that this result can not be valid again for general strongly pseudoconvex domains in higher dimensions. In fact, by Proposition 4.2, it is natural to expect that  $\lim_{z \rightarrow p} s_D(z) = 1$  for some  $p \in \partial D$  may imply, under a biholomorphic transformation,  $\partial D$  is spherical at  $p$ , where  $D$  is a strongly pseudoconvex domain in  $\mathbb{C}^n$  with  $n > 1$ . It is well known that a strongly pseudoconvex domain in  $\mathbb{C}^n$  ( $n > 1$ ) with analytic boundary which is spherical at some point is biholomorphic to the unit ball. Therefore it is natural to propose the following conjecture:

**Conjecture 4.1.** *Let  $D \subset \subset \mathbb{C}^n$  ( $n > 1$ ) be a strongly pseudoconvex domain with smooth boundary. If  $\lim_{z \rightarrow p} s_D(z) = 1$  for all  $p \in \partial D$ , or  $D$  has real analytic boundary and  $\lim_{z \rightarrow p} s_D(z) = 1$  for some  $p \in \partial D$ , then  $D$  is biholomorphic to the unit ball.*

## 5. APPLICATIONS

In this section, we use the results in the previous sections to study squeezing functions on some interesting domains, i.e., planar domains, Cartan-Hartogs domains, and a strongly pseudoconvex Reinhardt domain in  $\mathbb{C}^2$  that is not convex. As a consequence, other than recover some known facts, we obtain some new results about analytic and geometric properties of these domains.

**5.1. Planar domains.** For a planar domain  $D$  with smooth boundary, it was proved in [4] that  $\lim_{z \rightarrow p} s_D(z) = 1$  for all  $p \in \partial D$ . This result can be strengthened by using Theorem 4.4 as follows:

**Theorem 5.1.** *Let  $D$  be a bounded planar domain and  $p$  a smooth boundary point of  $D$ , then  $\lim_{z \rightarrow p} s_D(z) = 1$ .*

*Proof.* Let  $A$  be the connected component of  $\bar{\mathbb{C}} - D$  containing  $p$ , where  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . Then  $D' = \mathbb{C} - A$  is a simply connected domain in  $\mathbb{C}$  with  $p$  as a smooth boundary point. By Riemann's Mapping Theorem and Carathéodory's boundary correspondence (see e.g. [1]), there is a conformal map  $f : D' \rightarrow \Delta$  such that  $\lim_{z \rightarrow p} f(z) = q$  for some  $q$  in the unit circle. By definition, it is clear that  $B_{f(D)}(q) = 1$  and hence  $IB_D(p) = 1$ . By Theorem 4.4, we have  $\lim_{z \rightarrow p} s_D(z) = 1$ . This completes the proof of the theorem.  $\square$

Combing Theorem 5.1, Corollary 3.4, and Theorem 3.6, we get the following corollary, which seems already known in the literatures.

**Corollary 5.2.** *Let  $D$  be a planar domain smoothly bounded at  $p \in \partial D$ . Denote by  $\mathcal{H}_D^C, \mathcal{H}_D^K, \mathcal{H}_D^B$  the Carathéodory metric, the Kobayashi metric, and the Bergman metric on  $D$  respectively. Then we have*

$$\lim_{z \rightarrow p} \frac{\mathcal{H}_D^C(z)}{\mathcal{H}_D^K(z)} = 1,$$

and

$$\limsup_{z \rightarrow p} \frac{\mathcal{H}_D^B(z)}{\mathcal{H}_D^K(z)} \leq 8\pi.$$

*In particular, if  $D$  is smoothly bounded, then the above three intrinsic metrics on  $D$  are equivalent.*

**5.2. Cartan-Hartogs domains.** In this subsection, we investigate squeezing functions on Cartan-Hartogs domains, i.e., certain Hartogs domains based on classical bounded symmetric domains.

Recall that a classical bounded symmetric domain is a domain of one of the following four types:

$$\begin{aligned} D_I(r, s) &= \{Z = (z_{jk}) : I - Z\bar{Z}^t > 0, \text{ where } Z \text{ is an } r \times s \text{ matrix}\} \ (r \leq s), \\ D_{II}(p) &= \{Z = (z_{jk}) : I - Z\bar{Z}^t > 0, \text{ where } Z \text{ is a symmetric matrix of order } p\}, \\ D_{III}(q) &= \{Z = (z_{jk}) : I - Z\bar{Z}^t > 0, \text{ where } Z \text{ is a skew-symmetric matrix of order } q\}, \\ D_{IV}(n) &= \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n : 1 + |ZZ^t|^2 - 2Z\bar{Z}^t > 0, \ 1 - |ZZ^t| > 0\}. \end{aligned}$$

Let  $\Omega$  be a classical bounded symmetric domain, then the Cartan-Hartogs domain  $\hat{\Omega}_k$  associated to  $\Omega$  is defined to be

$$\hat{\Omega}_k = \{(Z, W) \in \Omega \times \mathbb{C}^m; \|W\|^2 < N(Z, Z)^k\},$$

where  $m$  is a positive integer and  $k$  is a positive real number,  $\|W\|$  is the standard Hermitian norm of  $W$ , and the generic norm  $N(Z, Z)$  for  $D_I(r, s)$ ,  $D_{II}(p)$ ,  $D_{III}(q)$ ,  $D_{IV}(n)$  are respectively  $\det(I - Z\bar{Z}^t)$ ,  $\det(I - Z\bar{Z}^t)$ ,  $\det(I + Z\bar{Z}^t)$ , and  $1 + |ZZ^t|^2 - 2Z\bar{Z}^t$ .

Cartan-Hartogs domains were introduced by W. Yin and G. Roos in 1998. In 1999, Yin computed the automorphism groups explicitly and gave the Bergman kernels and metrics of Cartan-Hatogs domains [20]. Yin and Zhang proved the four classical invariant metrics-the Carathéodoary metric, the Kobayashi metric, the Bergman metric and the Kähler-Einstein metric are all equivalent when the domains are convex [21]. Inspired by Liu-Sun-Yau's work [11], Yin proposed the following



open problem: whether Cartan-Hartogs domains are homogeneous regular [22]? In this subsection, we give an affirmative answer to this question. Consequently, by the work of Yeung in [19] and the results in §3, this leads to many nice analytic and geometric properties of Cartan-Hartogs domains. For example, these domains are hyperconvex and, with the Bergman and Kähler-Einstein metrics, have bounded geometry, and the four classical invariant metrics, as well as the volume forms considered in §3, on these domains are equivalent.

Let  $X : \Omega \times \mathbb{C} \rightarrow [0, 1)$  be defined by

$$X(Z, W) = \frac{\|W\|^2}{N(Z, Z)^k} - 1. \quad (12)$$

Then  $X$  is a defining function of  $\hat{\Omega}$  in  $\Omega \times \mathbb{C}^m$ .

**Theorem 5.3.** *For any positive number  $k$ , the Cartan-Hartogs domain  $\hat{\Omega}_k$  defined as above is a homogenous regular domain.*

*Proof.* We give the proof when  $\Omega = D_I(r, s)$  is a bounded symmetric domain of the first type in the above list. In this case,  $N(Z, Z) = \det(I - Z\bar{Z}^t)$ . Other cases can be proved with the same argument.

For any point  $(Z, W) \in \hat{\Omega}_k$ , it is known that there exists an automorphism  $f$  of  $\hat{\Omega}_k$  such that  $f(Z, W) = (0, \dots, 0, a)$  for some positive real number  $a$  (see [20]). So, by the holomorphic invariance and continuity of squeezing functions and Theorem 4.4, it suffices to prove that  $(0, \dots, 0, 1)$  is a g.s.c boundary point of  $\hat{\Omega}_k$  (see §§4.1 for definition).

We now compute the real Hessian  $\text{Hess}(X)(0, \dots, 0, 1)$  of the defining function  $X$  at  $(0, \dots, 0, 1)$ , where  $X(Z, W) = \frac{\|W\|^2}{N(Z, Z)^k} - 1$  as above. Let  $z_{jk} = x_{jk} + \sqrt{-1}y_{jk}$ ,  $1 \leq j \leq r, 1 \leq k \leq s$ , then  $\frac{\partial}{\partial x_{jk}} = \frac{\partial}{\partial z_{jk}} + \frac{\partial}{\partial \bar{z}_{jk}}$  and  $\frac{\partial}{\partial y_{jk}} = \sqrt{-1} \left( \frac{\partial}{\partial z_{jk}} - \frac{\partial}{\partial \bar{z}_{jk}} \right)$ . It is clear that  $\frac{\partial N}{\partial z_j}|_{z=0} = \frac{\partial N}{\partial \bar{z}_j}|_{z=0}$  and  $\frac{\partial^2 N}{\partial z_{jk} \partial \bar{z}_{lq}}|_{z=0} = \frac{\partial^2 N}{\partial \bar{z}_{jk} \partial z_{lq}}|_{z=0} = 0$  for all  $j, k, l, q$ . Note that  $dN(Z, Z) = N(Z, Z) \cdot \text{tr}((I - Z\bar{Z}^t)^{-1} d(I - Z\bar{Z}^t))$ . Direct calculations show that

$$\frac{\partial^2 N(z, z)}{\partial z_{jk} \partial \bar{z}_{lq}} \Big|_{z=0} = -\text{tr}(E_{jk} E_{lq}^t) = \begin{cases} -1, & j = l, k = q; \\ 0, & \text{otherwise,} \end{cases}$$

where  $E_{jk}$  denotes a  $(r \times s)$ -matrix whose components are non vanishing only at the  $(j, k)$  position. Therefore, we get

$$\text{Hess}(X)(0, \dots, 0, 1) = \begin{pmatrix} 2kI_{2rs} & 0 \\ 0 & 2I_{2m} \end{pmatrix}.$$

Note also that  $\nabla X(0, \dots, 0, 1) = 2\frac{\partial}{\partial u_m} \neq 0$ , where  $u_m$  is the real part of  $w_m$ , hence  $(0, \dots, 0, 1)$  is a strongly convex boundary point of  $\hat{\Omega}_k$ . On the other hand, it is clear that  $\hat{\Omega}_k \cap \{u_m = 1\} = \{(0, \dots, 0, 1)\}$ , so  $(0, \dots, 0, 1)$  is a g.s.c boundary point of  $\hat{\Omega}_k$ . This completes the proof of the theorem.  $\square$

For  $k$  tends to 0, the domains  $\hat{\Omega}_k$  increase to the product domain  $\Omega \times B^m$ . By Theorem 2.1, we have

$$\lim_{k \rightarrow 0} s_{\hat{\Omega}_k}(Z, W) = s_{\Omega \times B^m}(Z, W)$$

for all  $(Z, W) \in \hat{\Omega}_k$ . By Theorem 7.3 and Theorem 7.4 in [4], which are based on earlier work of Kubota in [10], we get

**Proposition 5.4.** *Let  $\hat{\Omega}_k$  as above, then*

$$\lim_{k \rightarrow 0} s_{\hat{\Omega}_k}(Z, W) = (s_{\Omega}^{-2} + 1)^{-1/2}$$

for all  $(Z, W) \in \hat{\Omega}_k$ , where  $s_{\Omega} = r^{-1/2}$ ,  $p^{-1/2}$ ,  $[q/2]^{-1/2}$ ,  $2^{-1/2}$  for  $\Omega = D_I(r, s)$ ,  $D_{II}(p)$ ,  $D_{III}(q)$  and  $D_{IV}(n)$ , respectively.

We now use Theorem 4.4 and the calculation in the proof of Theorem 5.3 to estimate the boundary behavior of squeezing functions of Thullen domains near the boundary point  $(1, 0)$ . Detailed estimate for general Cartan-Hartogs domains will be explored in a future work.

**Example 5.1.** *Let  $D_k = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^{2k} + |z_2|^2 < 1\}$ , where  $0 < k < 1$ . Then we have  $\liminf_{z \rightarrow (1, 0)} s_{D_k}(z) \geq \sqrt{k}$ .*

*Proof.* We choose  $\phi(z_1, z_2) := \frac{|z_1|^2}{(1 - |z_2|^2)^{1/k}} - 1$  as the defining function of  $D_k$ . By the calculation in the proof of Theorem 5.3, we see that  $\nabla\phi(1, 0) = 2\frac{\partial}{\partial x_1}$ , where  $x_1$  is the real part of  $z_1$ , and the real Hessian of  $\phi$  at  $(1, 0)$  is  $\begin{pmatrix} 2I_2 & 0 \\ 0 & \frac{2}{k}I_2 \end{pmatrix}$ . On the other hand, it is clear that  $e_{D_k}(1, 0) = 1$  (see §4.1 for notations). Hence  $B_{D_k}(1, 0) = k$  (see §4.1). By Theorem 4.4, we are done.  $\square$

**5.3. A Reinhardt domain.** The main aim of this subsection is to show that the Reinhardt domain defined in the following example is a homogenous regular domain. This domain is a strongly pseudoconvex domain with smooth boundary. Though we just consider a single domain here, it seems that the method can be generalized to study general strongly pseudoconvex domains.

**Example 5.2.** *Let  $D = \{(z_1, z_2) \in \mathbb{C}^2 \mid \rho(z_1, z_2) < 0\}$ , where  $\rho = \log^2 |z_1|^2 + \log^2 |z_2|^2 - 1$ . Then  $D$  is a homogenous regular domain.*

*Proof.* By Theorem 4.4, it suffices to prove  $IB_D(p) > 0$  for all  $p \in \partial D$ .

Let  $G = (S^1 \times S^1) \times \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ . Then  $G$  has a natural action on  $\mathbb{C}^* \times \mathbb{C}^*$  by holomorphic transformations generated by rotations, permutation of the coordinates, and maps given by  $(z_1, z_2) \mapsto (z_1^{-1}, z_2)$  and  $(z_1, z_2) \mapsto (z_1, z_2^{-1})$ . It is clear that  $D \subset \subset \mathbb{C}^* \times \mathbb{C}^*$  is stable under this action. Let  $A = \{(z_1, z_2) \mid e^{1/2} > |z_1| > 1, e^{1/2} > |z_2| > 1\} \cap \partial D$ . Then we have  $\partial D = G(A \cup \{(1, e^{1/2})\})$ .

We first prove that all points in  $A$  are g.s.c boundary points of  $D$ . The real Hessian of  $\rho$  is given by

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} \rho &= \frac{4(y_i^2 - x_i^2)}{(x_i^2 + y_i^2)^2} \log(x_i^2 + y_i^2) + \frac{8x_i^2}{(x_i^2 + y_i^2)^2}, \\ \frac{\partial^2}{\partial y_i^2} \rho &= \frac{4(x_i^2 - y_i^2)}{(x_i^2 + y_i^2)^2} \log(x_i^2 + y_i^2) + \frac{8y_i^2}{(x_i^2 + y_i^2)^2}, \\ \frac{\partial^2}{\partial x_i \partial y_i} \rho &= \frac{-8x_i y_i}{(x_i^2 + y_i^2)^2} \log(x_i^2 + y_i^2) + \frac{8x_i y_i}{(x_i^2 + y_i^2)^2}, \end{aligned} \tag{13}$$

where  $i = 1, 2$ . Since  $S^1 \times S^1$  acts on  $\mathbb{C}^2$  linearly, it suffices to consider points  $(z_1, z_2) \in A$  with  $y_1 = y_2 = 0$ . By the above calculation, it is clear that the real

Hessian of  $\rho$  at these points are positive definite. On the other hand,  $D$  in the domain  $\Omega$  given by

$$\Omega := D \cup \{|z_1| < 1, |z_2| < e^{1/2}\} \cup \{|z_1| < e^{1/2}, |z_2| < 1\},$$

which is a convex domain since it is Reinhardt and its intersection with  $\mathbb{R}^2$  is convex. Note that  $A$  consists of smooth boundary points of  $\Omega$ , so all points in  $A$  are g.s.c boundary points of  $D$ .

By the above calculations, we see that the Hessian of  $\rho$  is degenerate at the boundary point  $(1, e^{1/2})$ . But we will prove that there exists a biholomorphic map  $F : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}$  such that  $F(1, e^{1/2}) = (1, e^{1/2})$  is a g.s.c boundary point of  $F(D)$ . For  $\epsilon > 0$ , let  $F_\epsilon : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}$  be the biholomorphic map given by  $(z_1, z_2) \mapsto (z_1, z_2 + f_\epsilon(z_1))$ , where  $f_\epsilon(z_1) = \epsilon(z_1 + z_1^{-1} - 2) = \frac{\epsilon(z_1 - 1)^2}{z_1}$ . Then  $F_\epsilon(1, e^{1/2}) = (1, e^{1/2})$ . We can see  $F_\epsilon^{-1}(z_1, z_2) = (z_1, z_2 - f_\epsilon(z_1))$ . Let  $\tilde{\rho} = \rho \circ F_\epsilon^{-1}$  be the defining function of  $F_\epsilon(D)$ , then we have

$$\tilde{\rho}(z_1, z_2) = \log^2 |z_1|^2 + \log^2 |z_2 - f_\epsilon(z_1)|^2 - 1.$$

A direct calculation shows that

$$\begin{aligned} \frac{\partial^2}{\partial x_1'^2} \rho \circ F_\epsilon^{-1}|_{(1, e^{1/2})} &= 8 - 8e^{-1/2}\epsilon, \quad \frac{\partial^2}{\partial y_1'^2} \rho \circ F_\epsilon^{-1}|_{(1, e^{1/2})} = 8e^{-1/2}\epsilon, \\ \frac{\partial^2}{\partial x_2'^2} \rho \circ F_\epsilon^{-1}|_{(1, e^{1/2})} &= \frac{4}{e}, \quad \frac{\partial^2}{\partial y_2'^2} \rho \circ F_\epsilon^{-1}|_{(1, e^{1/2})} = \frac{4}{e}, \end{aligned} \quad (14)$$

and all other second order partial derivative of  $\tilde{\rho}$  at  $(1, e^{1/2})$  vanish. This implies  $(1, e^{1/2})$  is a strongly convex boundary point of  $F_\epsilon(D)$ . Note that  $\nabla \tilde{\rho}(1, e^{1/2}) = \nabla \rho(1, e^{1/2}) = 4e^{-1/2} \frac{\partial}{\partial x_2}$ . Hence, to prove  $(1, e^{1/2})$  is a g.s.c boundary point of  $F(D)$ , it suffices to prove  $\operatorname{Re}(z_2 + f_\epsilon(z_1)) < e^{1/2}$ , for all  $(z_1, z_2) \in \partial D - \{(1, e^{1/2})\}$ . Let  $z_i = e^{r_i/2} e^{\sqrt{-1}\theta_i}$ ,  $i = 1, 2$ , then, for  $z_1, z_2 \in \partial D$ , have

$$\begin{aligned} &\operatorname{Re}(z_2 + f_\epsilon(z_1)) \\ &= \epsilon((e^{r_1/2} + e^{-r_1/2}) \cos \theta_1 - 2) + e^{\frac{\sqrt{1-r_1^2}}{2}} \cos \theta_2 \\ &\leq \epsilon((e^{r_1/2} + e^{-r_1/2}) \cos \theta_1 - 2) + e^{\frac{\sqrt{1-r_1^2}}{2}}, \end{aligned}$$

and the equality holds only if  $\cos \theta_2 = 1$ . Let

$$g(r_1, \theta_1) = \epsilon((e^{r_1/2} + e^{-r_1/2}) \cos \theta_1 - 2) + e^{\frac{\sqrt{1-r_1^2}}{2}}.$$

Then, for  $\epsilon > 0$  small enough, a computation shows that  $\frac{\partial g}{\partial r_1} < 0$  for all  $r_1 \in (0, 1)$  and  $\theta_1 \in [0, 2\pi)$ . So we have

$$g(r_1, \theta_1) < g(0, \theta_1) = 2\epsilon(\cos \theta_1 - 1) + e^{-1/2}$$

for  $r_1 \in (0, 1)$ . Note also that  $g(1, \theta_1) \leq e^{-1/2}$  for  $\epsilon$  small enough, we have proved that  $\operatorname{Re}(z_1 + f_\epsilon(z_1)) \leq e^{-1/2}$  for  $(z_1, z_2) \in \partial D$ , and the equality holds if and only if  $(z_1, z_2) = (1, e^{1/2})$ . So  $(1, e^{1/2})$  is a g.s.c boundary point of  $F_\epsilon(D)$  for  $\epsilon$  small enough. This completes our proof.  $\square$

*Remark 5.1.* From the proof of the above example, we see that  $D$  is strongly pseudconvex. By [17], the automorphism group of  $D$  is compact. Since  $D$  is a Reinhardt domain that does not intersect the coordinate axis, this result can also

be seen in another way (see e.g. [23]). So  $D$  can not cover a compact complex manifold. On the other hand  $D$  is not convex. So it is not in the list of homogenous regular domains given in [19].

#### APPENDIX: PROOF OF PROPOSITION 4.1

The aim of the appendix is giving the proof of Proposition 4.1.

*Proof.* (Proof of Proposition 4.1) 1). It is clear that we just need to prove  $B_D$  is lower semi-continuous. For a point  $p \in \partial D$  with  $B_D(p) = 0$ , or equivalently  $e_D(p) = +\infty$ ,  $B_D$  is clearly lower semi-continuous at  $p$ .

Now we assume  $p \in \partial D$  and  $B_D(p) > 0$ . By 1) in Remark 4.1, it suffices to prove that  $e_D$  is upper semi-continuous at  $p$ .

For  $r > 0$  and  $q \in \partial D \cap U$ , let  $B_{q,r}$  be the ball defined by

$$|z - (q - r\nabla\rho(q))|^2 < r^2.$$

Let  $r > e_D(p)$  be fixed, we want to prove that, for some neighborhood  $V \subset U$  of  $p$ ,  $D \subset B_{q,r}$  for all  $q \in \partial D \cap V$ . Let

$$f_r(z, q) = \frac{|z - (q - r\nabla\rho(q))|^2 - r^2}{2r}.$$

By assumption, we can choose a local defining function  $\rho$  of  $D$  near  $p$  such that  $||\nabla\rho|| \equiv 1$  and  $Hess(\rho)(p) > cHess(f_r(z, p))|_{z=p}$  for some  $c > 1$ . By continuity, there is a neighborhood  $W$  of  $p$  such that

$$Hess(\rho)(q) > cHess(f_r(z, q))|_{z=q} \quad (15)$$

for  $q \in \partial D \cap W$ . We may assume  $W$  is convex and small enough. Then, for any fixed  $q \in \partial D \cap W$ , we have

$$\rho(z) = \Delta x \cdot \nabla\rho(q) + \sum_{i,j=1}^{2n} h_{i,j}(z, q) \Delta x_i \Delta x_j,$$

where  $\Delta x = (\Delta x_1, \dots, \Delta x_{2n}) = z - q$  is viewed as a vector in  $\mathbb{R}^{2n}$ . The key point here is that all  $h_{i,j}(z, q)$  are continuous on  $W \times (W \cap \partial D)$ , and  $h_{i,j}(q, q) = \frac{\partial^2 \rho}{\partial x_i \partial x_j}(q)$ . By (15), replacing  $W$  by a small enough relatively open subset of it, we have

$$\rho(z) - f_r(z, q) = \sum_{i,j=1}^{2n} h_{i,j}(z, q) \Delta x_i \Delta x_j - \Delta x Hess(f_r(z, q))|_{z=q} \Delta x^T > 0$$

for  $(z, q) \in W \times (W \cap \partial D)$ . This implies that  $W \subset B_{q,r}$  for all  $q \in \partial D \cap W$ .

On the other hand, it is clear that there is an open subset  $V$  of  $W$  such that  $D - W \subset B_{q,r}$  for all  $q \in \partial D \cap V$ . So, for all  $q \in \partial D \cap V$ , we have  $D \subset B_{q,r}$ . This implies  $e_D(q) \leq r$ . Let  $r \searrow e_D(p)$ , we see that  $e_D$  is upper semi-continuous at  $p$ .

2). Denote  $s \cdot e_D(p)$  by  $a$ . By similar argument as in the proof of 1), one can show that there is a neighborhood  $V$  of  $p$  in  $\mathbb{C}^n$  such that  $D \cap V \subset B_{q,a}$  for all  $q \in \partial D \cap V$ , where  $B_{q,a}$  is defined as in the proof of 1). So the proof of 2) is complete.  $\square$

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